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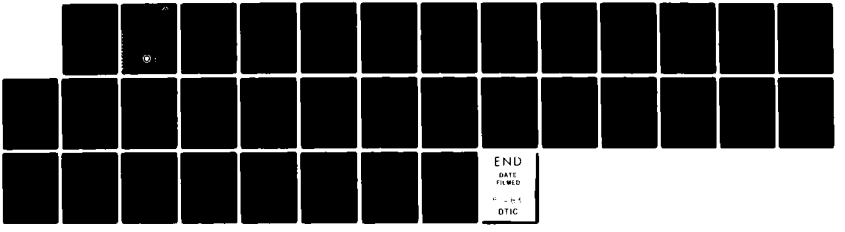

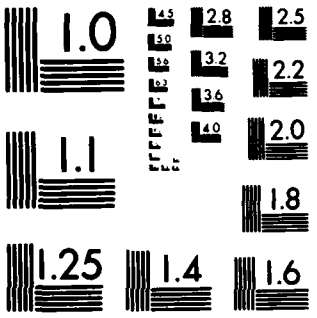

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Technical Report ICMA-83-52

ADAPTIVE FINITE ELEMENT PROCESSES IN STRUCTURAL MECHANICS*)

by

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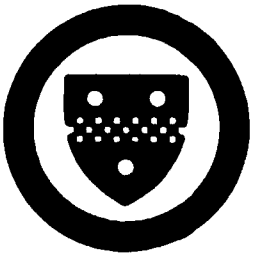

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ADAPTIVE FINITE ELEMENT PROCESSES IN STRUCTURAL MECHANICS*)

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ADAPTIVE FINITE ELEMENT PROCESSES IN STRUCTURAL MECHANICS*)

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1. INTRODUCTION

In general, engineering computations serve to analyze specific physical phenomena with the aim of reaching decisions about proposed designs; that is, for instance, the expected behavior of a construction, its safety features, and similar concerns. For a general assessment of the structure, qualitative results about the displacements, stress-distributions, etc. are needed with an engineering accuracy of, say 10-20%. On the other hand, the indicated decisions are based usually on relatively few data items with a higher accuracy, say, in the range of 2-5%, such as, the displacements or stresses in certain known critical points, stress-intensity factors, etc.

→ next page

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In order to achieve these diverse objectives the desired computational procedures should include among others the following features:

(i) Facilities for effective and reliable estimation of the errors of the computed results. These estimates should correspond to the specific aims of the particular computation.

(ii) Adaptively controlled computations, in order to achieve the desired error tolerances with a minimal cost and to reduce the preparatory work required of the user.

(iii) Post-processing facilities, which are designed to allow for the extraction of high-accuracy data items of the type needed for the critical decisions.

For the solution of classes of linear problems by means of finite-element approximations, algorithms for all three features are now beginning to be well understood. On the other hand, the nonlinear case is as yet not well developed and much still needs to be done before satisfactory algorithms for all three features can be constructed.

This paper is not intended to present an overview of the work in this area. Instead we address certain specific questions relating to the theoretical and computational aspects of the three features. In particular, the design of effective adaptive procedures requires a clear understanding of the objectives that are to be achieved. Accordingly, in Section 2 we discuss some new results about the concepts of feedback and adaptivity. In line with the summary nature of this paper,

these results are only illustrated for a one-dimensional, linear model problem. Then Section 3 addresses their extension to the more general two-dimensional, linear, elliptic class of problems which can be solved by the adaptive solver FEARS. In this generality, some of the results can be tested, as yet, only experimentally. In Section 4 we then turn to some of the post-processing approaches for linear problems which may meet the demands for the third of the indicated desirable features.

All the material in Sections 2 to 4 concerns linear problems. As noted before, for nonlinear problems, the corresponding theory is by far not so well developed. In part, this is due to the fact that the problem-formulations differ considerably between the two cases. In particular, in practical nonlinear problems we are faced usually with the need for considering parametrized systems of equations for which the solutions form manifolds in the space of the state and parameter variables. The problem then becomes the computational determination of the characteristic features of such solution manifolds. This is the topic of Section 5. For these parametrized equations it is not always readily apparent how to define the discretization errors needed for the first of our mentioned features. In Section 6 we present some recent results about a priori estimates of such discretization errors which in turn lend themselves to the development of the desired a posteriori estimates. Finally, in Section 7 we summarize some of the computational approaches which combine the trace of paths on the solution manifolds with mesh-refinement strategies and hence which may provide the basis for the

second of the cited facilities in the nonlinear case.

2. FEEDBACK AND ADAPTIVITY

The term "adaptation", or any of its cognates, is widely used today in scientific computing, but unfortunately it remains a rather ill-defined concept which is often used very loosely. Various definitions of adaptive processes have been given in the literature of such fields as automatic control theory (see eg. [22]) and artificial intelligence (see eg. [14]). A definition which applies directly to many of the procedures of numerical analysis is presented in [18]. Without entering into the details of the latter definition, we explain in this section the relevant concepts of feedback and adaptivity and summarize some of the possible theoretical results for the case of a simple model problem. The next section then addresses the extension of these results to more general problems.

Consider the two-point boundary value problem

$$L[u] \equiv -\frac{d}{dx}a(x)\frac{du}{dx} + b(x)u = f(x),\ x \in I = (0,1), \quad (2.1)$$

$$u(0) = u(1) = 0 \quad (2.2)$$

where

$$0 < \underline{\alpha} \le a(x) \le \bar{\alpha},\ 0 \le b(x) \le \beta,\ x \in \bar{I}. \quad (2.3)$$

As usual, we define on $\overset{\circ}{H}{}^1(I) \times \overset{\circ}{H}{}^1(I)$ the bilinear form

$$B(u,v) = \int_I (au'v' + buv)dx \quad (2.4)$$

and note that the norm on $H^1(I)$ is equivalent with the

energy norm $||u||_E = (B(u,u))^{1/2}$. The weak solution $u_o \in \overset{\circ}{H}{}^1(I)$ of (2.1)/(2.2) is defined by

$$B(u_o,v) = F(v) \equiv \int_I fvdx, \quad \forall\ v \in H^1(I). \tag{2.5}$$

Let $\mathcal{F}$ be a family of finite-dimensional subspaces $S \subset \overset{\circ}{H}{}^1(I)$. For any $S \in \mathcal{F}$ the approximate solution $u(S) \in S$ of (2.1)/(2.2) is specified by

$$B(u(S),v) = F(v), \quad \forall\ v \in S \tag{2.6}$$

and hence the optimal error achievable with subspaces of $\mathcal{F}$ of a fixed dimension $d > 0$ is

$$\phi(u_o,d) = \inf\{||u_o-u(S)||_E;\ S \in \mathcal{F},\ \dim S = d\}. \tag{2.7}$$

For example, we obtain a set $\mathcal{F} = \mathcal{F}_p$ by considering all meshes

$$\Delta:\ 0 = x_o^\Delta < x_1^o < \ldots < x_{m(x)}^\Delta = 1 \tag{2.8}$$

and specifying $S \in \mathcal{F}_p$ as the space of all C^o-functions on I which are polynomials of degree at most r on each subinterval $I_i = [x_{i-1}^\Delta, x_i^\Delta]$, $i = 1,\ldots,m(\Delta)$, of Δ. This selection leads to the classical h-version of the finite element method for (2.1)/(2.2) with elements of order r. If we use instead a fixed mesh but vary the order of the polynomial pieces separately on each subinterval, then we arrive at the p-version of the method (see eg. [8]).

A feedback system for (2.1)/(2.2) on some given set $\mathcal{F}$ may be characterized as a process which produces sequences (or trajectories) of subspaces $\{S_k\}_1^\infty \subset \mathcal{F}$. This is accomplished

by some transition function τ which produces for any index (time) k the successor $S_{k+1} \in F$ on the basis of information about the sequence up to k.

For a precise definition of this concept we refer to [18]. Here we present only an example of a simple transition function $\tau = \tau_p$ for the case $F = F_p$. For any $S = S(\Delta) \in F_p$ consider for each i, $1 \leq i \leq m(\Delta)$, the exact (weak) solutions z_i of (2.1) on $I_i^\Delta = [x_{i-1}^\Delta, x_i^\Delta]$ subject to the boundary conditions $z_i(x_{i-1}^\Delta) = u(S)(x_{i-1}^\Delta)$, $z_i(x_i^\Delta) = u(S)(x_i^\Delta)$. Then the error indicators

$$\eta_i^\Delta = ||z_i - u(S(\Delta))||_{E,I_i^\Delta}, \quad i = 1,\ldots,m(\Delta), \tag{2.9}$$

can be computed approximately or estimated from above and below in terms of the residuals $r_i^\Delta = L[u(S)] - f$ on I_i^Δ. Moreover

$$\varepsilon(\Delta) = [\sum_{i=1}^{m(\Delta)} (\eta_i^\Delta)^2]^{1/2} \tag{2.10}$$

represents an estimator for the error $||u_o - u(S)||_E$, (see [4], [5], [6]). Now a simple mesh-refinement strategy constructs *from* Δ *a new mesh* Δ' *by halving the subinterval* I_i^Δ for which η_i^Δ is maximal. This corresponds to the transition function $\tau = \tau_p$ which produces as the successor $\tau(S(\Delta))$ of the current space $S(\Delta) \in F_p$ the space $S' = S(\Delta') \in F_p$.

Now any feedback system on a set F will be called an adaptive system if it is optimal with respect to some performance measure, or, more precisely, if under this measure its performance is not worse than that of any other feedback system.

Generally, a performance measure is a function μ which associates with any trajectory $\{S_k\} \subset F$ produced by a feedback system an element $\mu\{S_k\}$ of a given partially order set M. In the simplest case we might use the set $M = \{0,1\}$ which allows us to distinguish only between good (=0) or bad (=1) performance. Two typical performance measures of this type may be defined as follows:

$$\mu\{S_k\} = \begin{cases} 0 \text{ if } \lim\limits_{k\to\infty} ||u_o - u(S_k)||_E = 0 \\ \\ 1 \text{ otherwise} \end{cases} \tag{2.11}$$

$$\mu\{S_k\} = \begin{cases} 0 \text{ if } ||u_o - u(S_k)||_E \leq C\ \phi(u_o, \dim(S_k)), \forall\, k \geq 0 \\ \\ 1 \text{ otherwise} \end{cases} \tag{2.12}$$

where in (2.12) the constant C depends on the problem and the solution u_o, but not on k.

These and other measures are discussed in [1], [2], [9]. We mention here only a few partial results for our model problem phrased not necessarily in their most general form:

Theorem 2.1: For the problem (2.1)-(2.3) and any $u_o \in \overset{\circ}{H}{}^1(I)$ consider the simple feedback system on $F = F_p$ defined by the transition function τ_p. Then the resulting trajectories $\{S_k\}$ satisfy $\lim\limits_{k\to\infty} ||u_o - u(S_k)||_E = 0$ and hence the system is adaptive with respect to the performance measure (2.11).

For the performance measure (2.12) more restrictive conditions are required. In essence, we compare here the

convergence rate for the computed sequence of meshes with an "ideal" rate. For this we need to know -- irrespective of any feedback system -- the existence of a sequence of meshes for which some comparison with the ideal rate applies. We call this a comparison sequence; its existence is a condition on the solution $u_o \in \overset{\circ}{H}^1(I)$ which we wish to approximate. In essence, a theorem of the following form can then be proved (see [9]).

Theorem 2.2: Suppose that the coefficients a,b of (2.1/3) are sufficiently smooth and that for the given $u_o \in \overset{\circ}{H}^1(I)$ there exists a comparison sequence of meshes with certain properties. Then, for the feedback system on F_p defined by τ_p, the resulting trajectories satisfy

$$||u_o - u(S_k)||_E \leq C\ \phi(u_o, \dim S(u_k)), \quad k \geq 0$$

with a constant that depends on the problem and the comparison sequence, but not on k. Hence, the feedback system is adaptive under the performance measure (2.12).

The result implies that under the particular conditions the rate of convergence is independent of the smoothness of the solution. The conditions of the theorem are not overly stringent. For example, they hold for $u_o = x^\alpha - x \in \overset{\circ}{H}^1(I)$, $\alpha > \frac{1}{2}$, in which case the rate of convergence is of the order of $(\dim S(\Delta_k))^{-r}$ independent of α, (see [9]). We refer also to [5] for related results.

The simple transition function τ_p used in both theorems is computationally costly since it modifies the meshes very little from one t step. Moreover, in practice we can work

only with a finite segment $S_1,\ldots,S_n$ of any trajectory. In that case, it is reasonable to demand that the computation of the final approximate solution $u(S_n)$ is at most, say, twice or thrice the cummulative cost of the computation up to that point.

For this we have to introduce an appropriate stopping criterion which, obviously, needs to be based on an error estimator such as (2.10). The construction of such a stopping rule is suggested by the following two results:

Theorem 2.3: For $u_o \in \overset{\circ}{H}{}^1(I)$ and under the condition (2.3) we have

$$C_2\, \varepsilon(\Delta) \leq ||u_o - u(S(\Delta))||_E \leq C_1\, \varepsilon(\Delta)$$

where the constants $C_1, C_2 > 0$ are independent of $S(\Delta)$.

Theorem 2.4: Under the conditions of theorem 2.2 we have

$$\lim_{k\to\infty} \frac{\varepsilon(\Delta_k)}{||u_o - u(S(\Delta_k))||_E} = 1.$$

For some results about the adaptivity of feedback systems with transition functions incorporating such stopping criteria see [1], [9]. It should be noted also that the results indicated here remain valid for finite sets of right-hand sides f in (2.1).

3. MORE GENERAL PROBLEMS AND THE FEARS SYSTEM

The results sketched in the previous section allow

extensions to more general problems in two space dimensions. In particular, results presented in [2] address the family of problems and feedback approaches upon which the FEARS-program [Finite Element Adaptive Research Solver] is based.

For a detailed description of FEARS we refer to [16] and [23]. In brief, FEARS solves certain systems of elliptic equations in two space dimensions and permits also combinations with one-dimensional problems, as, for instance, those needed for modelling stiffeners in elasticity problems. Curved quadrilateral elements of first order are used. As in (2.9) error indicators are associated with each element and from them an estimator (2.10) of the solution-error is obtained. For details about these a posteriori estimates see [2].

FEARS utilizes various modes of mesh-refinement. In particular, a so-called long pass involves the recomputation of the solution on the new mesh, while in a short pass no such recomputation is performed but instead decisions about further refinements are based on an extrapolation of the refinement patterns observed in previous meshes. Thus, by combining these short and long passes in various ways different feedback mechanisms are obtained.

Theorem 2.1 carries over, in full generality, to trajectories generated by long passes in FEARS, and the same is true for Theorem 2.3 (see [3], [4]). On the other hand, the analogue of Theorem 2.2 is not available in sufficient generality. Yet, all numerical experiments performed with FEARS suggest that such a theorem should be valid also in this case. Theorem 2.4 does extend, however, under certain additional assumptions about the meshes (see [2]). By

appropriate use of the short passes it is possible to ensure that the cost of the computation of the solution on the final meshes of the refinement sequence is proportional to the cumulative cost of the computation up to that point. In fact, experience shows that the proportionality factor is of the order of 1.5 to 3.0.

As an illustration we present here an example of the typical performance of FEARS. More specifically we consider the problem $\Delta u = 0$ on the domain shown in Figure 1 with the indicated boundary conditions. The figure shows also the dotted

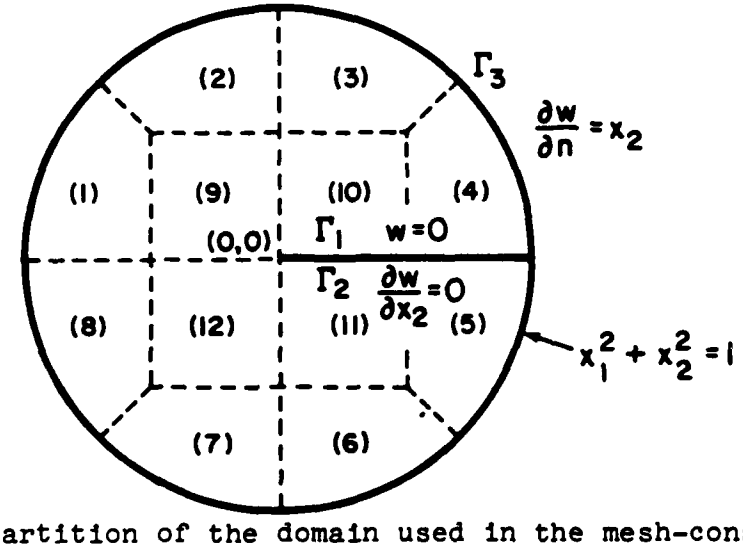

Figure 1

partition of the domain used in the mesh-construction of FEARS (see [23]). The solution has a strong singularity at the origin of the type $r^{1/4}$; that is, the solution belongs to $H^{5/4-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Hence for sequences of uniform meshes the rate of convergence is of the order of $N^{-1/8}$ with respect to the degrees of freedom N. This is certainly a slow rate.

Figure 2 depicts the achieved accuracy -- measured in the energy norm -- as function of the number of elements N^* ($\sim N$). More specifically, the true errors are given for a sequence (a) of uniform meshes, (b) of adaptively constructed meshes

based on the combination of long (L) and short (S) passes LSLSSLSSSLSSSSLSSSL, and (c) of adaptively constructed meshes using long passes only. In addition, the computed error estimates are shown. All data are in percent of the square root of the exact energy

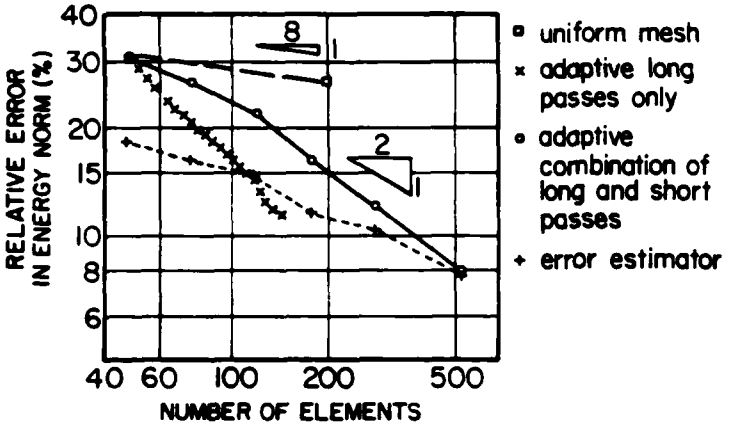

Figure 2

The figure indicates that the procedure is convergent in accordance with the two-dimensional analogue of Theorem 2.1. The rate of convergence for the adaptively constructed meshes has the maximum rate $N^{-1/2}$, which confirms experimentally the analogue of Theorem 2.2. Actually we see a slightly better rate because the higher errors are here principally confined to a neighborhood of the origin. On the other hand, the rate of convergence for the uniform meshes is very close to $N^{-1/8}$. In fact, the accuracy of 8.5% obtained by the adaptive mesh of 537 elements would require a uniform mesh of more than 10^6 elements.

The adaptive meshes based on long passes give better results than those involving a combination of passes. In dependence of N this is certainly expected. The situation is different in Figure 3 where the errors are shown in relation to the total machine time. Here the effectiveness of the combination of long and short passes shows up very clearly.

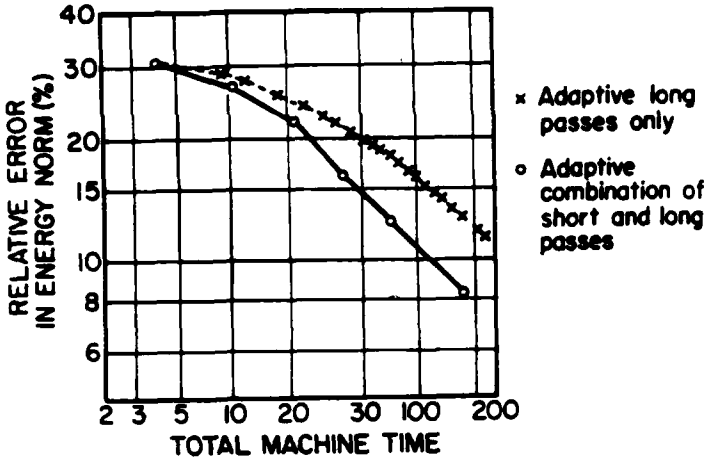

Figure 3

Figure 4 gives the earlier mentioned factor of the cost of the final long pass to the total time for the sequence of long and short passes. Evidently the factor is of the order of two.

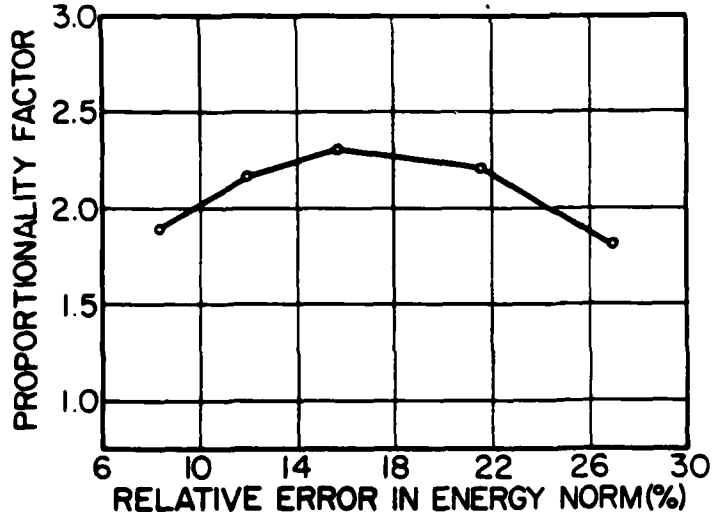

Figure 4

Finally, Figure 5 shows the effectivity index θ = est. error/true error as a function of the achieved accuracy. Experience shows that often, but not always, the estimated errors are smaller than the true errors. But for the accuracies of 10% or better the effectivity index exceeds 0.9. In line with this, experience has shown that the stopping

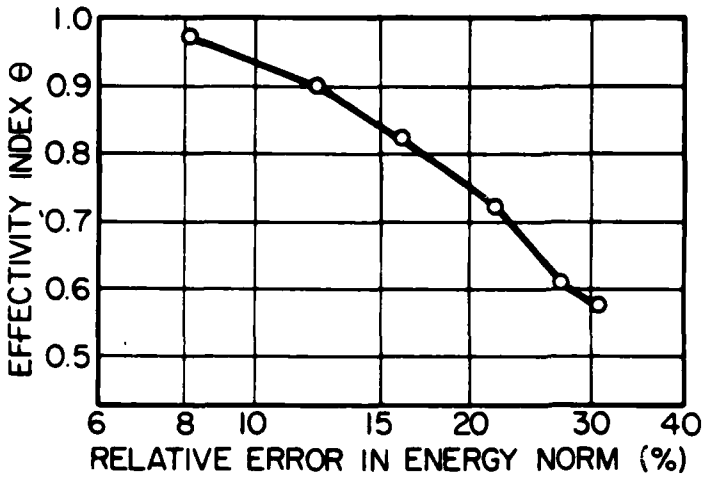

Figure 5

criterion based on the estimators is very reliable. Generally, the effectivity index is better for smoother solutions; it is also better for the procedure based only on long passes.

The results discussed here for this single example are consistent with and typical of all the experimental evidence we have obtained so far with FEARS.

4. POST-PROCESSING

As mentioned in the Introduction, in practical engineering computations important decisions are usually based on relatively few, but highly accurate data items. A very effective approach to the computation of such data is the application of post-processing techniques to the computer solutions. For a detailed study of this approach we refer to [3]. Here we sketch only the principal ideas for the simple

model problem

$$-\Delta u = 1, \quad \text{on} \quad \Omega = (0,1) \times (0,1) \tag{4.1}$$

$$u = 0, \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

Suppose that we are interested in the values $\phi_1 = u(0,0)$ and $\phi_2 = \frac{\partial u}{\partial x}(1,0)$. The main idea is to write

$$\phi_i = \int_\Omega u_o \xi_i dx + R_i, \quad i = 1,2 \tag{4.3}$$

where u_o again denotes the exact solution of (4.1)/(4.2) and the R_i are integrals that may be computed from the input data. For instance, for ϕ_1 we may use

$$\xi_1 = -\Delta\psi, \quad R_1 = \int_\Omega \psi dx$$

$$\psi(x) = \frac{1}{2\pi}\,(1u||x||_2 - 1u[\tfrac{1}{2}(1+x_1^2)(1+x_2^2)]^{1/2}).$$

Once (4.3) is available, an approximate value of $\tilde{\phi}_i$ can be computed by replacing in (4.3) the exact solution u_o with the computed solution $u(S)$ (see the notation of Section 2). Then the error can be written in the form

$$\tilde{\phi}_i - \phi_i = B(u(S)-u_o, \psi_i(S)-\psi_{oi})$$

where

$$B(u,v) = \int_\Omega [\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}]dx$$

and ψ_{oi} is the solution of $-\Delta\psi = \xi_i$ subject to the boundary conditions (4.2) while $\psi_i(S)$ is the corresponding

approximate solution. The computation of $\psi_1(S)$ can be done simultaneously with the computation of $u(S)$ at no great additional expense.

Table 2 shows a comparison of the results for ϕ_1 and ϕ_2 obtained both from the computed solution $u(S)$ and the post-processing approach.

No. of elements in quarter domain	4	16	64
Energy norm error of $u(S)$	30.1%	15.2%	7.6%
Error in ϕ_1 computed from $u(S)$	5.4%	1.3%	.31%
Error in ϕ_1 computed by post-processing	2.5%	.63%	.16%
Error in ϕ_2 computed from $u(S)$	29%	16%	8.7%
Error in ϕ_2 computed from post-processing	1.3%	.32%	.076%

Table 2

Table 3 gives the effectivity index for the error estimates of the computed values of ϕ_1 and ϕ_2; that is, the quotient of the estimated error divided by the exact error.

No. of elements in quarter domain	4	16	64
Effectivity index for ϕ_1	1.07	.98	.99
Effectivity index for ϕ_2	.93	1.02	1.06

Table 3

5. PROBLEM FORMULATIONS IN THE NONLINEAR CASE

By far the majority of todays finite-element computations involve linear problems; but, in recent years, the nonlinear analysis of structural systems has become increasingly important in engineering applications. Not unexpectedly, there are many fundamental differences between linear and nonlinear problems, and, in fact, the problem formulations for the latter show many special features not present in the linear case.

First and foremost, it has to be recognized that the theoretical approaches and computationally techniques depend strongly on the type and properties of the nonlinearities. In structural problems we may distinguish four principal sources of nonlinearities, namely (see [11])

(i) geometric nonlinearities, due to nonlinear strain-displacement relations

(ii) material nonlinearities, due to nonlinear constitutive equations

(iii) force nonlinearities, due to nonlinear stress boundary conditions

(iv) kinematic constraint nonlinearities, due to nonlinear displacement boundary conditions.

The source of nonlinearity affects the form of the resulting nonlinear equations and hence the solution approaches. The corresponding numerical problems are as yet not completely understood. This is particularly true for the nonlinearities of type (ii) to (iv).

As indicated in the Introduction, engineering computations serve to predict the behavior of a physical system under study so as to reach decisions about proposed designs. In the case of structural problems, the term "behavior" refers, say, to deformations in response to the action of external influence quantities, such as loads, changes in material properties and geometrical data, etc. This means that the equations of the system also depend on a set of influence parameters and the objective is to assess the behavior of the solutions under variations of the parameters. In the linear case, this variation is also assumed to be linear and we need only compute a few specific solutions to achieve our objective. In nonlinear problems, however, the computation of a few specific solutions provides little or no insight into the behavior of the system and we are led to consider the set of all solutions depending on all the parameters in a specific range.

Mathematically, this means that we are faced with parametrized equations of the form

$$F(y,\lambda) = b \qquad (5.1)$$

where y varies in a space Y of state (behavior) variables, λ in a parameter space Λ, F is a given mapping from $Y \times \Lambda$ into Y, and $b \in Y$ a prescribed element in the range of F. In general, the set of all solutions $(y,\lambda) \in Y \times \Lambda$ of (5.1) forms a manifold in $Y \times \Lambda$. In structural mechanics, this set is now usually called the equilibrium manifold of the structure. The objective then becomes a computational analysis of the form and characteristic features of this manifold.

Clearly, the specific features to be determined for a particular problem depend on the objectives of the computation. Generally, interest centers on computing solutions on the manifold along specified paths which represent, for instance, particular loading-regimes, etc. Then, the boundaries of the stability regions on the manifold are to be explored, and the form of the manifold near specific singularities is to be determined.

Besides these general objectives, there are various specific solution data that are needed for certain decisions about the structure under consideration. A typical example of this is the so-called question about conservative input data. In general, the external input data influencing a structure are only known within certain ranges; for instance, this is certainly true for the loads to which the structure will be subjected during its lifetime. Thus, for the design it is essential to know the values of the parameters which represent worst possible cases. These are called conservative input data. Their determination represents a search problem on the solution manifold.

As an example, consider a clamped, thin, shallow circular arch which has been used as a test case by various authors (see eg. [15], [21]). Let U and W be the radial and axial displacements, R the arch-radius, A its cross-sectional area, h the thickness, and E Young's modulus. With the dimensionless displacements $u = U/h$, $w = W/h$, the total potential energy -- non-dimensionalized by dividing by $EAR(h/R)^2$ -- is given by

$$\frac{1}{2}\int_{-\theta_o}^{\theta_o}[(\frac{dw}{d\theta}-u)+\frac{1}{2}\frac{h}{R}(\frac{du}{d\theta})^2]^2d\theta+\frac{\alpha_1}{2}\int_{-\theta_o}^{\theta_o}(\frac{d^2u}{d\theta^2})^2d\theta-\alpha_2\int_{-\theta_o}^{+\theta_o}pud\theta.$$

Here $p = p(\theta)$ is the dimensionless radial load and α_1,α_2 are dimensionless constants. Each end is assumed to be pinned; that is, we have the boundary conditions

$$u(\pm\theta_o) = 0, \quad w(\pm\theta_o) = 0, \quad \frac{d^2u}{d\theta^2}(\pm\theta_o) = 0.$$

The finite element approximations introduced in [25] were applied. More specifically, we used a uniform mesh with eight elements, $\theta_o = 15^o$, and the constants $\alpha_1 = 3.8716 \times 10^{-6}$, $\alpha_2 = 8.2752 \times 10^{-2}$ corresponding to data in [15].

Suppose that the following two-parameter family of loads is to be considered

$$p(\mu,\nu)(\theta) = \begin{cases} (1-\frac{4}{\theta_o}(\nu-\theta))\mu, & \text{for } \max(-\theta_o,\nu-\frac{\theta_o}{4}) \le \theta \le \nu \\ (1-\frac{4}{\theta_o}(\theta-\nu))\mu, & \text{for } \nu \le \theta \le \min(\theta_o,\nu+\frac{\theta_o}{4}) \\ 0, & \text{otherwise.} \end{cases}$$

Thus the load is a piecewise-linear hat-function which has the value μ at $\theta = \nu$ and is zero outside the interval centered at ν of width $\theta_o/2$. For any fixed position ν the path on the manifold determined by varying μ encounters a first limit point which represents a point where the arch is expected to buckle. Figure 6 shows the variation of this buckling load with the value of ν. We see that the most dangerous loads occur about for $\nu = \pm 0.16\theta_o$; hence, these values represent the desired conservative data. The computations were performed with the limit-point facilities of the continuation package PITCON (see [19]).

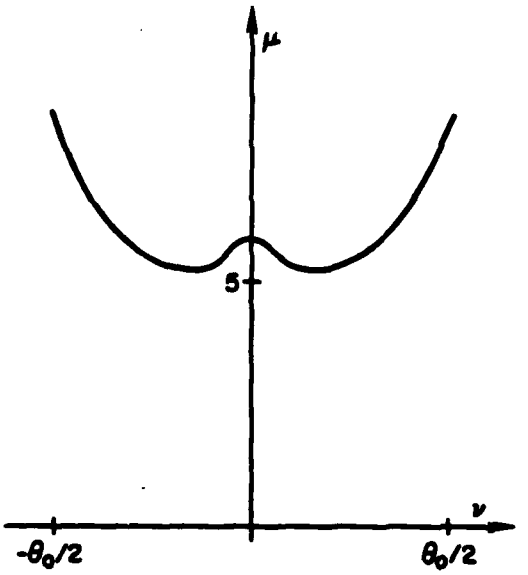

Figure 6

6. ERROR ESTIMATES

As in the linear case, a desirable feature for nonlinear structural computations is the estimation of the errors of the

computed results. But, not unexpectedly, for nonlinear problems the situation is considerably more difficult and much remains to be done. In fact, for the solutions of the parametrized equations of the previous section, it is not always readily apparent how to define the error. In fact, if, say, M denotes the equilibrium manifold of a given infinite-dimensional equation (5.1) and $\tilde{M}$ is the solution manifold of a finite-dimensional approximation of this equation, then it is not obvious how to relate points of M and $\tilde{M}$. Frequently, the parameter space Λ of (5.1) is finite-dimensional and hence there is no need to discretize $\lambda \in \Lambda$. In that case, we might wish to compare points of M and $\tilde{M}$ corresponding to the same values of λ. But already simple examples show that for specific λ there may exist a point only on one of the two manifolds.

In order to resolve this problem it is necessary to consider first the equation of a priori estimates for parametrized equations. For mildly nonlinear elliptic boundary value problems and one-dimensional parameter spaces such estimates were first developed in [10]. A different approach was used in [12] to provide similar results for more general boundary value problems. In [13] this approach was extended to the case of any finite-dimensional parameter space. We present here only a brief summary of the results in [12], [13].

Let X,Y be Banach-spaces and $F: S \subset X \to Y$ a mapping of class $C^r(S)$, $r \geq 1$, on an open connected subset S of X. Suppose that F is a Fredholm operator of index $m \geq 1$ on S and that the regularity set $R(F) = \{x \in S;\ \text{rge } DF(x) = Y\}$ is non-empty. Then $R(F)$ is an open subset of X and for any

regular value $b \in F(R(F))$ the regular solution set

$$M(b) = \{x \in R(F);\ F(x) = b\}$$

in an m-dimensional C^r-manifold in X. For proofs we refer to [12].

For any theoretical and computational study of $M(b)$ we need to specify which local parametrizations of the manifold we will use. At a point $x_o \in M(b)$ a local parametrization of $M(b)$ shall be a triple $\{V,A,T\}$ consisting of

(i) a closed subspace V of X such that $V \cap \ker DF(x_o) = \{0\}$,

(ii) an isomorphism $A \in L(Y,V)$ from Y onto V, and

(iii) an m-dimensional subspace T of X such that $X = V \oplus T$.

The meaning of this concept is contained in the following result (see [12]):

Theorem 6.1: Under the stated conditions for F and $M(b)$, let $\{V,A,T\}$ be a local parametrization of the manifold at $x_o \in M(b)$. Then there exists an open ball $B \subset T$ with $0 \in B$, an open neighborhood $U \subset X$ of x_o, and a unique C^r-function $\eta: B \to Y$ such that

$$M(b) \cap U = \{x \in X;\ x = x_o + t + A\eta(t),\ \forall\, t \in B\}.$$

The geometric interpretation is sketched in the composite Figure 7. In many applications certain parameters are identified explicitly; that is, one has a natural splitting $X = W \oplus Z$, $\dim Z = m$, where W is isomorphic with the range

space Y. The question then arises at what points $x_o \in M(b)$ this natural splitting gives rise to a local parametrization; that is, when the given parameters can be used to parametrize the manifold near x_o. Clearly, the condition here is $W \cap \ker DF(x_o) = \{0\}$ and this holds exactly if $\operatorname{rge} D_W F(x_o) = Y$. Then $\{W, D_W F(x_o)^{-1}, Z\}$ indeed is a local parametrization of $M(b)$ at x_o. Such points are called nonsingular points of $M(b)$ with respect to the natural parameters.

At points $x_o \in M(b)$ where $W \cap \ker DF(x_o) \neq \{0\}$ the natural parameters cannot be used to parametrize $M(b)$. Such points can be characterized by the dimension of this intersection. The simplest case arises when $\dim(W \cap \ker DF(x_o)) = 1$; these are limit points with respect to the direction of the intersection. For more details we refer to [13].

We turn now to a discretization of our problem. Once again, it is useful to assume the existence of a natural splitting $X = W \oplus Z$, $\dim Z = m$, where W is isomorphic with Y. More specifically, we assume that there is an operator Q such that

$$Q \in L(X,Y), \quad Z = \ker Q, \quad \operatorname{rge} Q|W = Y.$$

Often we have $F: S \subset Y \times R^m \to Y$ in which case Q can be taken as the natural projection from $Y \times R^m$ onto Y.

The natural splitting has to be considered since in such applications only the state space W is discretized and the parameters are left untouched. Suppose now that the discretization is defined by a family of linear maps $P_h \in L(Y)$ of finite rank indexed by a positive $h > 0$ such that $\lim_{h \to 0} P_h y = y$ for all $y \in Y$. Then the finite-dimensional sub-

spaces

$$Y_h = P_h Y \subset Y, \quad W_h = (Q|W)^{-1} Y_h \subset W, \quad X_h = W_h \oplus Z \subset X$$

are well-defined and we may consider the approximating problem

$$F_h(x) = y_{oh}, \quad x \in R(F_h), \quad y_{oh} = P_h y_o \tag{6.1}$$

$$F_h: S_h \subset X_h \to Y_h, \quad F_h(x) = P_h F(x), \quad x \in S_h = S \cap X_h.$$

In order to compare the solution manifolds of these discretized problems with $M(b)$ we need to assume that the approximation is sufficiently close to ensure that the same local parametrization can be used on these manifolds, (see Figure 7). As usual, a stability condition is required to ensure the convergence. Without elaborating upon the specific details of this condition, we sketch here only the general form of the resulting theorems (see [12] and [13]):

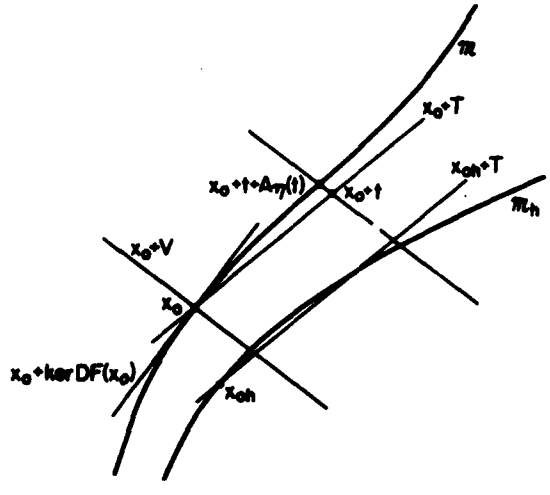

Figure 7

Theorem 6.2: Under the stated conditions about F and $M(b)$, let $\{V,A,T\}$ be a local parametrization of $M(b)$ at $x_o \in M(b)$ and suppose that the indicated stability condition holds. Then, for all sufficiently small $h > 0$, the approxi-

mate problems (6.1) possess solutions $x_{oh} \in R(F_h)$ for which $\lim_{h\to 0} x_{oh} = x_o$.

Theorem 6.3: In addition to the conditions of Theorem 6.2, let DF be Lipschitz-continuous on bounded subsets of $S \subset X$. Denote by $x: B \subset T \to S$ the representation of $M(b)$ near x_o defined on the open ball $B \subset T$, $0 \in B$. Then there exists a closed ball $B_o \subset B$, $0 \in B_o$ such that for all sufficiently small $h > 0$ the approximate problems have solutions $x_h: B_o \to S_h$ for which

$$||x(t)-x_h(t)|| \leq C||(I-P_h)Qx(t)||, \quad t \in B_o \tag{6.2}$$

where the constant C is independent of h and t.

As a typical example consider the mildly nonlinear problem in the weak form

$$a(u,w) + \int_\Omega g(u,\lambda)w\,d\xi = 0, \quad w \in \overset{\circ}{H}{}^1(\Omega), \quad \lambda \in R^m,$$

where Ω is a suitable bounded domain in R^n, and the bilinear form

$$a(u,w) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial u}{\partial \xi_j} \frac{\partial w}{\partial \xi_i}\, d\xi, \quad u,w \in \overset{\circ}{H}{}^1(\Omega)$$

has sufficiently smooth coefficients and is strongly elliptic. Then the operator $K: L^2(\Omega) \to H^1(\Omega)$ defined by $a(Ku,w) = (u,w)_o$, $u \in L^2(\Omega)$, $w \in \overset{\circ}{H}{}^1(\Omega)$, is compact and the problem is equivalent with

$$u + KG(u,\lambda) = y_o$$

where $G: \overset{\circ}{H}^1(\Omega) \times R^m \to L^2(\Omega)$ is defined by the C^r-map g. It turns out that this is a Fredholm map of index m and that, with Q as the natural projection from $\overset{\circ}{H}^1(\Omega) \times R^m$ onto $\overset{\circ}{H}^1(\Omega)$, the mentioned stability condition is satisfies and Theorems 6.2 and 6.3 apply.

On the basis of a priori estimates of the form (6.2) we can now develop a posteriori error estimates along the same lines as for the linear problems. Some examples for this are presented in [7] and, as in the linear case, the effectivity of these estimates turns out to be consistently high. However, the theory of these a posteriori estimates for nonlinear problems is not, as yet, very well developed and, in particular, such results as Theorem 2.3 are still lacking. In [7] it was shown also that a posteriori estimates can be constructed for buckling loads and the location of limit points.

7. CONTINUATION PROCESSES

For the computational analysis of the solution manifolds considered here the basic methods are general continuation processes for the trace of specified paths on the manifold. If the problem formulation involves natural parameters, then these paths are defined usually by combinations of the parameter values with one degree of freedom. But interest may center also on paths which are specified implicitly, for example, as paths in the critical boundary of the problem (see eg. [17]).

The literature on continuation methods is rather large. For a survey relating to structural mechanics we refer only to

[20] where also further references may be found. No attempt shall be made to present here the details of these methods. Broadly speaking a typical such process (PITCON, [19]) involves the following steps:

(1) Compute the (normalized) tangent vector at the current point on the path.

(2) Determine a new local parametrization of the path near this point.

(3) Determine a steplength.

(4) Use the steplength to compute a predicted point further along the path.

(5) Start a corrector iteration from the predicted point to obtain a new point (approximately) on the path.

The overall process uses extensive feedback as the basis of the various decisions. This is particularly true for the above steps (2), (3), and (5). The feedback mechanisms used here are often rather complex in nature, and this raises the question whether the overall process can be shown to be adaptive in the sense of Section 2. Up to now, no such results have been formulated, although, there appears to be the possibility of proving at least some partial results. There is certainly a need for more studies of this area.

On the basis of the estimates of the previous section it turns out that the discretization errors along any continuation path often vary considerably. On the other hand, our objectives usually require that these errors be maintained reasonably well within a tolerance interval. This suggests again the use of mesh-refinements (and de-refinements) in

conjunction with the continuation process.

In its simplest form such a combined process begins with a mesh that meets a prescribed error-tolerance requirement. Then, during the continuation process, the error estimators are monitored and, at any computed point where these estimators exceed the upper tolerance a mesh-refinement strategy is applied to meet the tolerance requirement. Similarly, we may work also with a lower error tolerance and de-refine the mesh whenever the estimates fall below that lower value.

Figure 8 indicates the effect of such a strategy for a problem involving a nonlinear rod. We shall not present here the details of the problem-formulation but refer, for that, to [7]. In brief, the indicated values of N refer to the number of quadratic C^0-elements that were used during the corresponding segments of the path. The error tolerance was 2%. Since the continuation process takes relatively large steps along the path, there is no guarantee that between these points the error tolerance does not exceed the required tolerance. A more sophisticated control-strategy may alleviate the situation somewhat. At the same time, problems of this type cause difficulties in developing theoretical results of the type of Theorems 2.1 and 2.2 for the combined continuation and mesh-refinement processes. The level of expected difficulty of such results for nonlinear problems suggests that at first some emphasis should be placed upon carefully designed experiments which confirm or refute various conjectures about the adaptivity of the feedback mechanisms that are used in these settings.

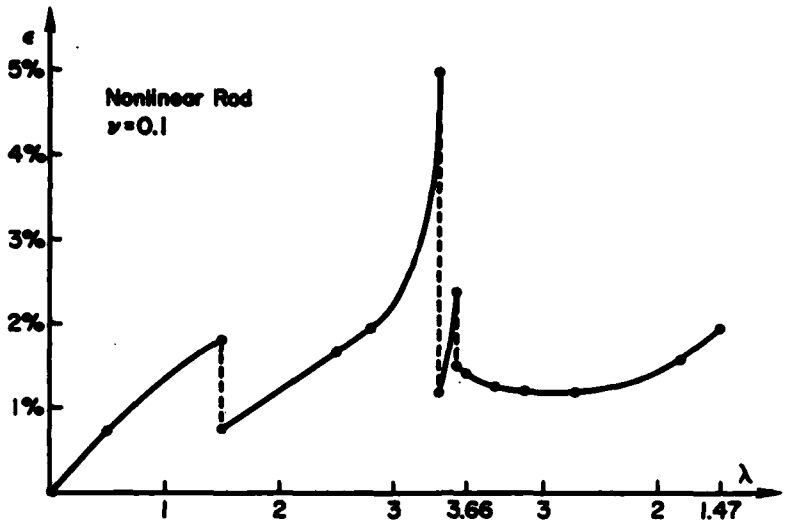

Figure 8

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